Preprint

A characterization of function $\pi(x)$ in relation to twin primes

By

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Abstract

It's still a conjecture the Euclid's statement about the existence of infinitely many twin primes. It is not known if it is true but a strong evidence seems to justify the conjecture. In 2004 Arensdorf proposed a proof of the conjecture but unfortunately a serious error was found in the proof.

Conditions are known instead, in order to prove that a pair (p-2, p) is a pair of twin primes.

In this study a specific property of the prime-counting function $\pi(x)$ *is provided: we will prove that each pair of primes (p, p+2) (independently if infinitely many or not) is such that the function* $\pi(p)$ *is equal to the summation of the difference of the fractional parts of the ratios of p+2 and p+1 to each prime lower than or equal to* p.

The result lends itself well to processing further applications and insights.

Key words: *twin primes, characterization, prime-counting function* $\pi(x)$*, twin prime conjecture.*

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Notation

In addition to the symbols commonly used:

$\left\lfloor \frac{x}{y} \right\rfloor$ = floor function of $\frac{x}{y}$, $y \neq 0$

P = set of primes

$I°(n) = \{i \ : \ \text{odd, with } 3 \leq i \leq n\}$,

$\left\{ \frac{x}{y} \right\}$ = fractional part function of $\frac{x}{y}$, $y \neq 0$

Introduction

Two primes are *twin primes* if their difference is 2. Conditions are known in order to prove that a pair (p-2, p) is a pair of twin primes.

In 1949 **[4]** P.A. Clement proved that *integers n, n+2 are a pair of twin primes if and only if:*

$$4\,[(n-1)! + 1] \equiv -n \quad (\text{mod } n(n+2))$$

In 1963 **[7]** F. Pellegrino proved the following theorem, derived from Wilson's theorem **[3]** **[10]**:

Two natural numbers p-2 e p, with $p \geq 5$, are both primes if and only if:

$$4\left[\frac{(p-3)!}{p-2}\right] \equiv -5 \quad (mod.\,p)$$

In 2004 S.M. Ruiz **[8]** proved as a corollary of the main theorem concerning integers, that:

For odd n > 7, the pair (p, p+2) of integers are twin primes if and only if:

$$\sum_{i\ odd}^{j}\left(\left\lfloor\frac{p+2}{i}\right\rfloor - \left\lfloor\frac{p+1}{i}\right\rfloor + \left\lfloor\frac{p}{i}\right\rfloor - \left\lfloor\frac{p-1}{i}\right\rfloor\right) = 2$$

where the summation is over odd values of i through $j = \lfloor p/3 \rfloor$

The attempts over the past decades[1], to solve the twin prime conjecture on the basis of these and many other approaches **[2]** **[5]** proved to be unsuccessful.

In 2004 Arensdorf **[1]** proposed a proof of the conjecture but a serious error was found in the proof (in particular Lemma 8 was found to be incorrect).

At the same time a strong numerical evidence[2] seems to justify the conjecture.

In this paper is presented an interesting property of the prime-counting function π(x) **[10]** and its relation to twin primes.

[1] *e.g.* the primality tests **[3],** the Sieve methods **[10],** the Rabin-Miller **[6]** probabilistic algorithm for testing primality etc.

[2] Two distributed computing project, Twin Prime Search (TPS) and Prime Grid search for large twin primes and have produced many record-largest twin primes

A basic property of natural numbers

We recall the following property **[3] [8] [9]** seldom mentioned in standard texts, of which we provide a full demonstration:

Necessary and sufficient condition for:

$$(1)\quad \left\lfloor \frac{n}{i} \right\rfloor - \left\lfloor \frac{n-1}{i} \right\rfloor = 0$$

is that i does not divide $n \in N$, $n > 0$ *with* $i \in N$ $i \neq 1$ $i < n$

Proof

If i does not divide $n \in N$ then $\left\lfloor \frac{n}{i} \right\rfloor - \left\lfloor \frac{n-1}{i} \right\rfloor = 0$ in fact:

$n = k_1\, i + q_1$ with $i > q_1$

with $k_1 = \left\lfloor \frac{n}{i} \right\rfloor$ floor function of $\frac{n}{i}$ and q_1= reminder ($q_1 > 0$ by assumption)

since $q_1 > 0$ then:

$n - 1 = k_1\, i + q_2$ with q_2= reminder, $q_2 \geq 0$

hence $k_1 = k_2$ *i.e.* $\left\lfloor \frac{n}{i} \right\rfloor = \left\lfloor \frac{n-1}{i} \right\rfloor$

At the same time:

If $\left\lfloor \frac{n}{i} \right\rfloor - \left\lfloor \frac{n-1}{i} \right\rfloor = 0$ then i does not divide $n \in N$

in fact:

$n - 1 = \left\lfloor \frac{n-1}{i} \right\rfloor \cdot i + q_2$ with q_2= reminder, $q_2 \geq 0$

by assumption $\left\lfloor \frac{n}{i} \right\rfloor = \left\lfloor \frac{n-1}{i} \right\rfloor$

then:

$n = \left\lfloor \frac{n}{i} \right\rfloor \cdot i + q_1$ with $q_1 > 0$

Since $q_1 > 0$ then i does not divide $n \in N$.

The statement (1) follows. ■

Now we establish one theorem which will become useful in proving Theorem 2: it can be derived from the A/M corollary **[8].** We provide an independent demonstration.

Theorem 1

Let p ∈ P then p+2 ∈ P if and only if:

$$(2)\ \sum_i \left(\left\lfloor\frac{p+2}{i}\right\rfloor - \left\lfloor\frac{p+1}{i}\right\rfloor\right) = 0 \ \ \forall\, i \in I°(p)$$

<u>*Proof*</u>

First part:

We establish the equivalence of:

$$(2)\ \sum_i \left(\left\lfloor\frac{p+2}{i}\right\rfloor - \left\lfloor\frac{p+1}{i}\right\rfloor\right) = 0 \ \ \forall\, i \in I°(p)$$

And

$$(3)\ \left\lfloor\frac{p+2}{i}\right\rfloor - \left\lfloor\frac{p+1}{i}\right\rfloor = 0 \ \ \forall\, i \in I°(p)$$

In fact: if $\sum_i \left(\left\lfloor\frac{p+2}{i}\right\rfloor - \left\lfloor\frac{p+1}{i}\right\rfloor\right) = 0 \ \ \forall \in I°(p)$

Then: $\left\lfloor\frac{p+2}{i}\right\rfloor - \left\lfloor\frac{p+1}{i}\right\rfloor = 0 \ \ \forall \in I°(p)$

because each term of the summation is greater than or equal to zero.

And if: $\left\lfloor\frac{p+2}{i}\right\rfloor - \left\lfloor\frac{p+1}{i}\right\rfloor = 0 \ \ \forall \in I°(p)$

Then: $\sum_i \left(\left\lfloor\frac{p+2}{i}\right\rfloor - \left\lfloor\frac{p+1}{i}\right\rfloor\right) = 0 \ \ \forall \in I°(p)$

Second part:

We establish that:

If *p+2* ∈ P then for all natural numbers *i*: $1 < i \leq p$:

$$(3)\ \left\lfloor\frac{p+2}{i}\right\rfloor - \left\lfloor\frac{p+1}{i}\right\rfloor = 0$$

As proved (1) (by definition of prime number) any prime *p+2* verifies (3) and consequently the statement (2).

At the same time:

If: $\left\lfloor\frac{p+2}{i}\right\rfloor - \left\lfloor\frac{p+1}{i}\right\rfloor = 0 \ \ \forall\, i \in I°(p)$ then p+2 ∈ P

Because *p+2* is not divisible by any odd number between 3 and *p* and at the same time it's not divisible by any even number since *p+2* is odd. ■

Theorem 2

Let p a prime such that (p, p+2) is a pair of primes, then:

$$(4)\quad \pi(p) = \sum_{2}^{p}\left(\left\{\frac{p+2}{\bar{p}_i}\right\} - \left\{\frac{p+1}{\bar{p}_i}\right\}\right) \qquad \textit{for each prime } \bar{p}_i \leq p \in P$$

First part:

We establish that eq. (3) leads to:

$$(5)\ \left\{\frac{p+2}{i}\right\} - \left\{\frac{p+1}{i}\right\} = 1 \quad \forall\, i \in I°(p)$$

Proof of the first part

Let *p* ∈ P and consider the necessary and sufficient condition for having *p, p+2* ∈ P (3):

$$(3)\ \left\lfloor\frac{p+2}{i}\right\rfloor - \left\lfloor\frac{p+1}{i}\right\rfloor = 0 \quad \forall\, i \in I°(p)$$

We observe that:

$p+2=k_1\, i + q_1$ with $i > q_1 > 0$

$p+1=k_2\, i + q_2$ with $i > q_2 \geq 0$

It is necessary and sufficient condition for having *p+2* ∈ P , to have $k_1 = k_2$ (3).

It follows that:

$q_2 - q_1 = 1$ *i.e.* for any *p* ∈ P such that *p+2* ∈ P, holds:

$$(5)\ \left\{\frac{p+2}{i}\right\} - \left\{\frac{p+1}{i}\right\} = 1$$

We have established that for p ∈ P then p+2 ∈ P if and only if the difference of fractional parts (5) calculated for any odd number between 3 and n, is equal to 1.

Second part:

Eq. (3) and eq. (5), can be restricted to the primes $\bar{p} \in I°(p)$, *i.e.* to the primes between *3* and *p.*

Proof of the second part

If $\left\lfloor\frac{p+2}{i}\right\rfloor - \left\lfloor\frac{p+1}{i}\right\rfloor = 0 \quad \forall\, i \in I°(p) \quad i.e.$ for each odd $i \in I°(p)$

Obviously the same holds for each prime $\bar{p} \in I°(p)$:

$$\left\lfloor\frac{p+2}{\bar{p}}\right\rfloor - \left\lfloor\frac{p+1}{\bar{p}}\right\rfloor = 0$$

At the same time:

Let $\bar{p} \in I°(p)$, if:

$$(6)\ \left\lfloor\frac{p+2}{\bar{p}}\right\rfloor - \left\lfloor\frac{p+1}{\bar{p}}\right\rfloor = 0$$

Each of his multiple $\bar{p}\,n$ lower than (or equal[3]) to p verifies the same condition:

(7) $\left\lfloor \frac{p+2}{\bar{p}\,n} \right\rfloor - \left\lfloor \frac{p+1}{\bar{p}\,n} \right\rfloor = 0 \quad with\ \bar{p} \cdot n\ < p$

In fact:

$\bar{k}_1 = \frac{p+2-\bar{q}_1}{n \cdot \bar{p}}$ and $\bar{k}_2 = \frac{p+1-\bar{q}_2}{n \cdot \bar{p}}$

$\bar{k}_1 = \frac{k_1 \cdot \bar{p} + q_1 - \bar{q}_1}{n \cdot \bar{p}}$ and $\bar{k}_2 = \frac{k_2 \cdot \bar{p} + q_2 - \bar{q}_2}{n \cdot \bar{p}}$

Hence:

$\bar{k}_1 = \frac{k_1 \cdot \bar{p}}{n \cdot \bar{p}} + \frac{q_1 - \bar{q}_1}{n \cdot \bar{p}}$ e $\bar{k}_2 = \frac{k_2 \cdot \bar{p}}{n \cdot \bar{p}} + \frac{q_2 - \bar{q}_2}{n \cdot \bar{p}}$

But $k_1 = k_2$ (by assumption (6))

hence $q_1 - \bar{q}_1 = q_2 - \bar{q}_2$

and since $q_1 - q_2 = 1$ from eq. (5) it follows that: $\bar{q}_1 - \bar{q}_2 = 1$

Hence, if $k_1 = k_2$ then $\bar{k}_1 = \bar{k}_2$.

Hence:

if each number $i \in$ I°(p) verifies eq. (7), the same condition (6) obviously holds for each $\bar{p} \in$ I°(p)

and vice versa:

if $\bar{p}$ verifies eq. (6), the same condition (7) holds for each composite number $\bar{p} \cdot n \in$ I°(p) and since:

$\{\bar{p} \in \text{I°(p)}\} \cup \{\bar{p} \cdot n \in \text{I°(p)}\} \supset \text{I°(p)}$

We have established that for p ∈ P, necessary and sufficient condition for $p+2 \in$ P is:

$$(8)\ \left\{\frac{p+2}{\bar{p}}\right\} - \left\{\frac{p+1}{\bar{p}}\right\} = 1 \quad \text{for each prime } 3 \le \bar{p} \le p$$

Third Part:

The case with $\bar{p}_i = 2$ is the same of each prime $\bar{p}$: $3 \le \bar{p} \le p$.

Proof of the third part

Since number 2 doesn't divide $p+2$ (odd number) we have (1):

$\left\lfloor \frac{p+2}{2} \right\rfloor - \left\lfloor \frac{p+1}{2} \right\rfloor = 0$

And the same procedure used in the first part for eq. (3) to eq. (5) leads to:

$$(8b)\ \left\{\frac{p+2}{2}\right\} - \left\{\frac{p+1}{2}\right\} = 1$$

[3] Of course it doesn't exist any prime $\bar{p} \cdot n = p$ with $\bar{p}, p \in P$

Findings of Theorem 2:

We have established that for $p \in$ P such that p+2 ∈ P the difference of the fractional parts (8) (8b) calculated for each prime $\overline{p}_i$ lower than or equal to *p*, is always equal to 1.

It follows that the summation (4) of the fractional parts 'counts' exactly the number of primes lower than or equal to a given prime p ∈ *P* such that *p+2* ∈ P.

In other words Theorem 2 establishes that the number of primes lower than or equal to a given prime *p* such that *p+2* ∈ P is the summation (4) of the difference of the fractional parts of the ratios of *p+2* ∈ *P* e *p+1* to each prime lower than or equal to *p*. ■

An equivalent way of expressing the statement of Theorem 2:

The statement (4) of Theorem 2 can also be expressed as follows:

Let p a prime such that (p, p+2) is a pair of primes, then:

$$\pi(p) = \sum_2^p (a_i - b_i)$$

With a_i and b_i such that:

$$a_i = \left\{\frac{p+2}{\bar{p}_i}\right\} \;\; and \;\; b_i = \left\{\frac{p+1}{\bar{p}_i}\right\} \quad for\ each\ prime\ \bar{p}_i \le \ p \in P$$

Example

For p=29 the pair of twin primes is (29, 31)

	Difference of fractional parts
1	0
2	1 (8b)
3	1 (8)
5	1 (8)
7	1 (8)
9	1
11	1 (8)
13	1 (8)
15	1
17	1 (8)
19	1 (8)
21	1
23	1 (8)
25	1
27	1
29	1 (8)

It's evident that counting the primes lower than or equal to p = 29 is counting the fractional parts according to (8b) and (8)[4].

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[4] At the same time we observe that every composite number (*e.g.* n= 9) verifies the same condition (8) (in this case of n=3)

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